DISPLAY SYSTEMS *are in the past!* TANGENT DISPLAY MAPS *are the future!*

Dalhousie UNIVERSITY

Marcello Lanfranchi

joint work with

Geoff Cruttwell

A pullback problem

The pullback of two smooth maps might not be smooth.

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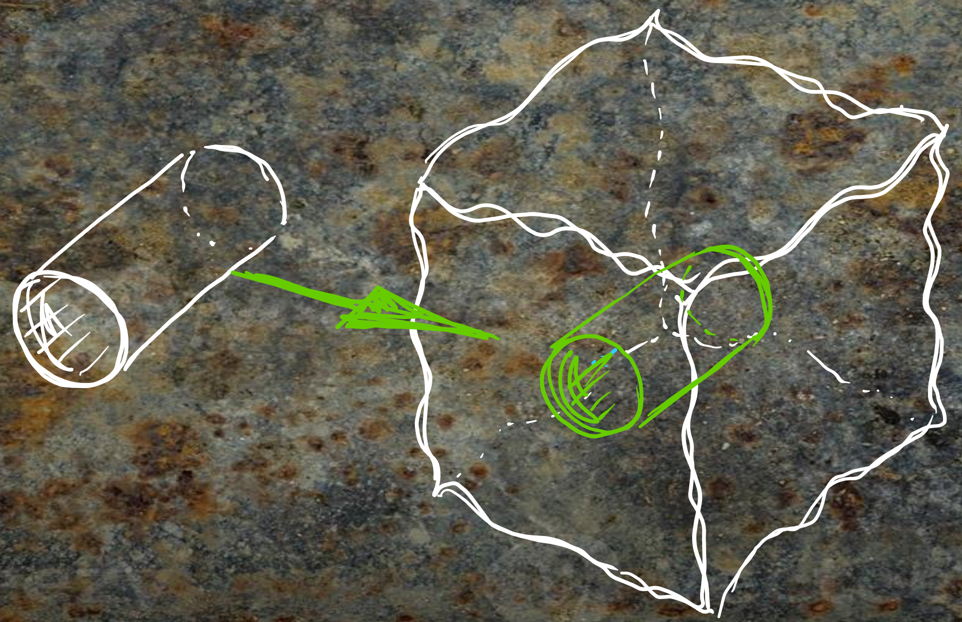

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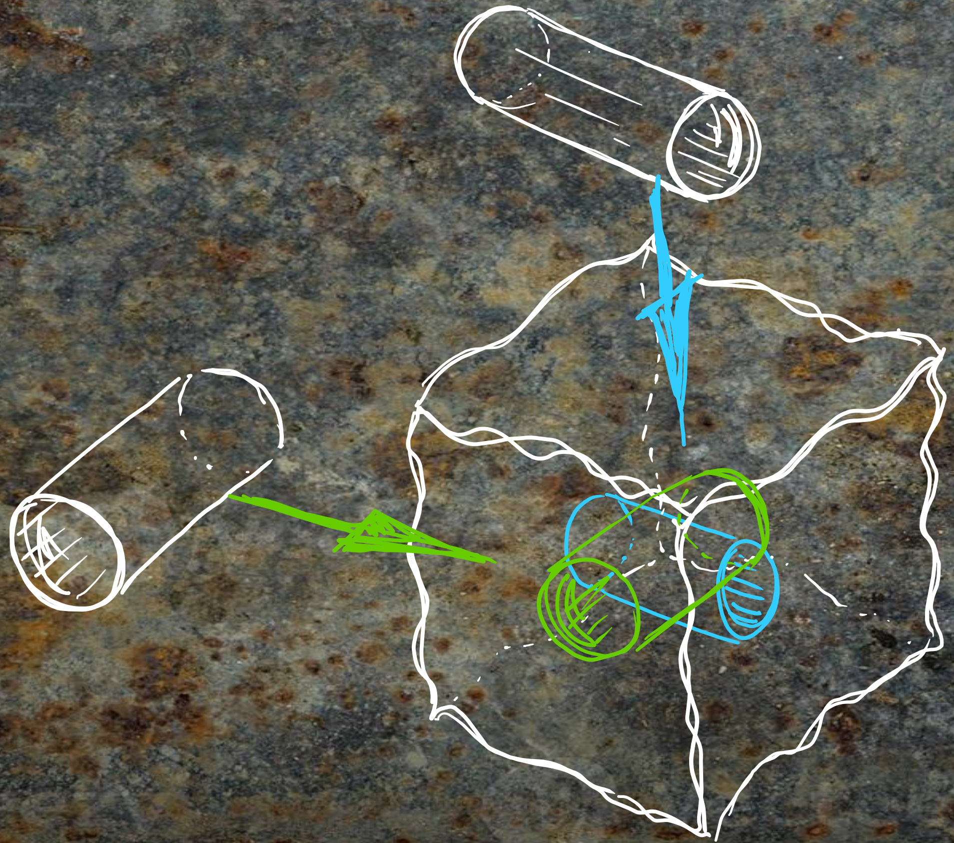

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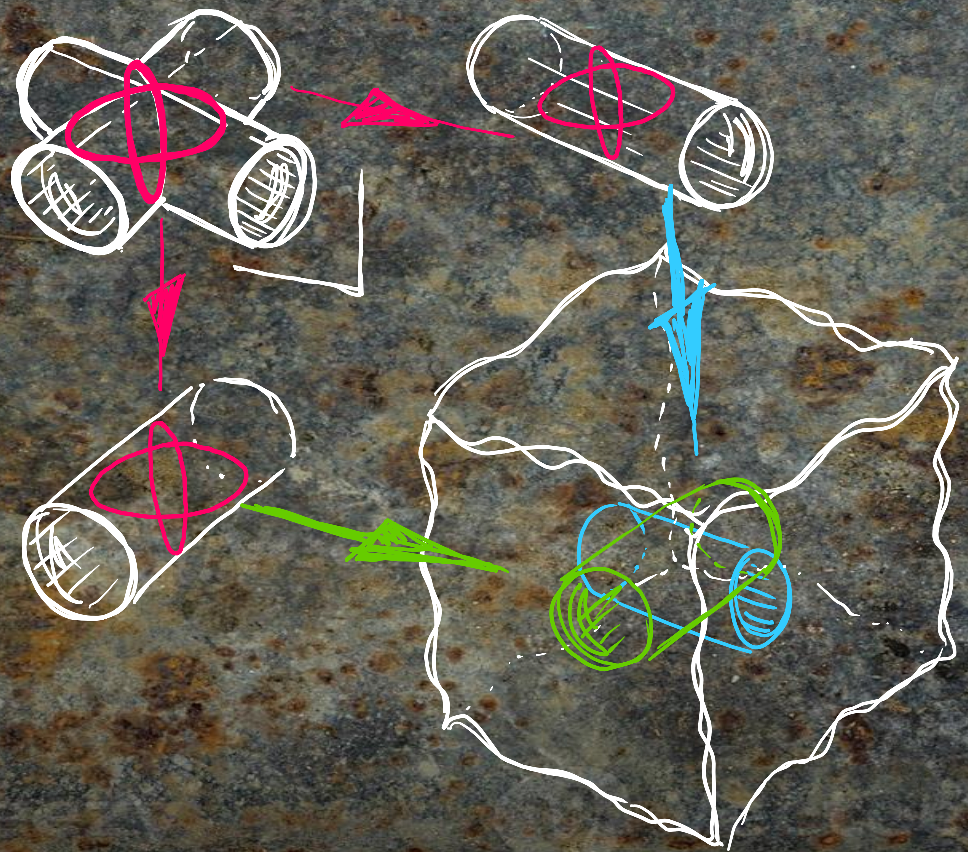
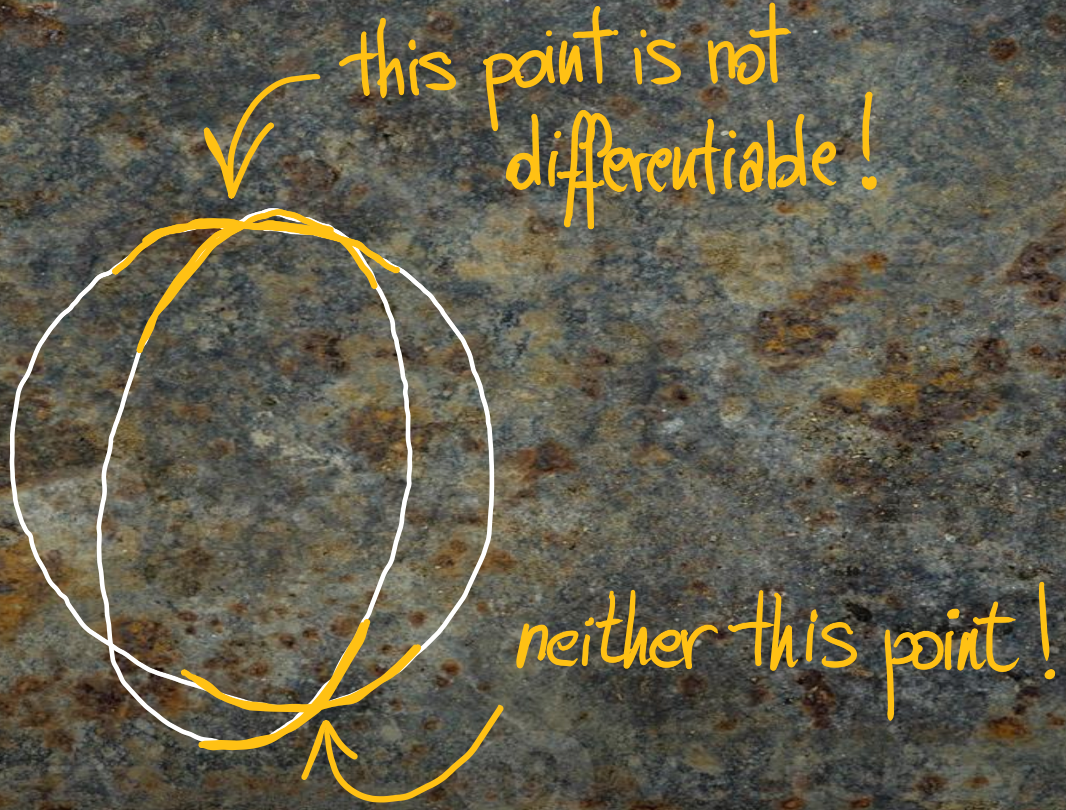


A pullback problem

The pullback of two smooth maps might not be smooth.

So, in differential geometry one needs to be careful with pullbacks.



A pullback problem

This is why we do not require a tangent category to have all pullbacks.

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A tangent category is a category whose objects have a notion of local linearity and maps are locally linear.

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this encodes local linearity

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For connections!

Imagine that you want to move a vector rigidly.

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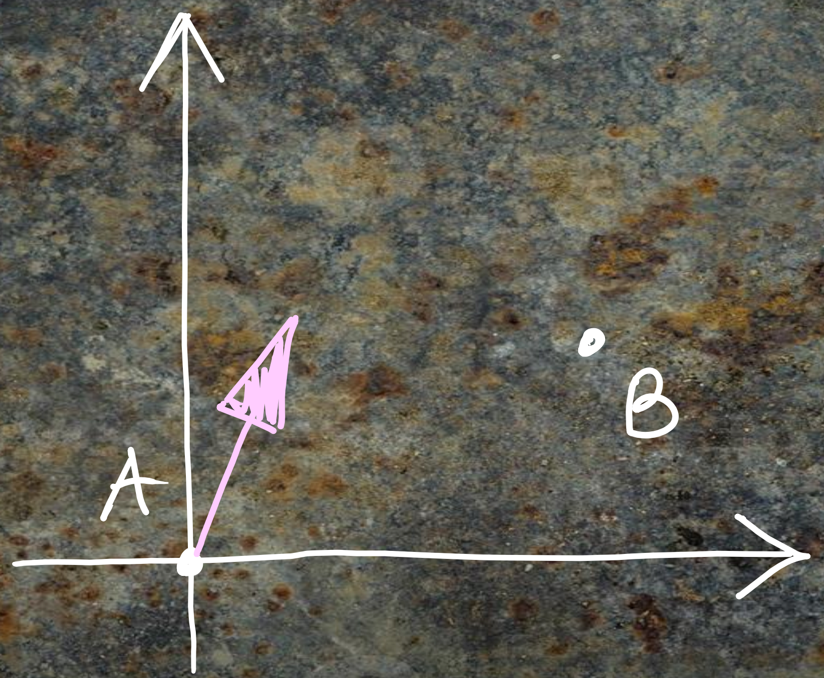
On the Cartesian plane is easy.



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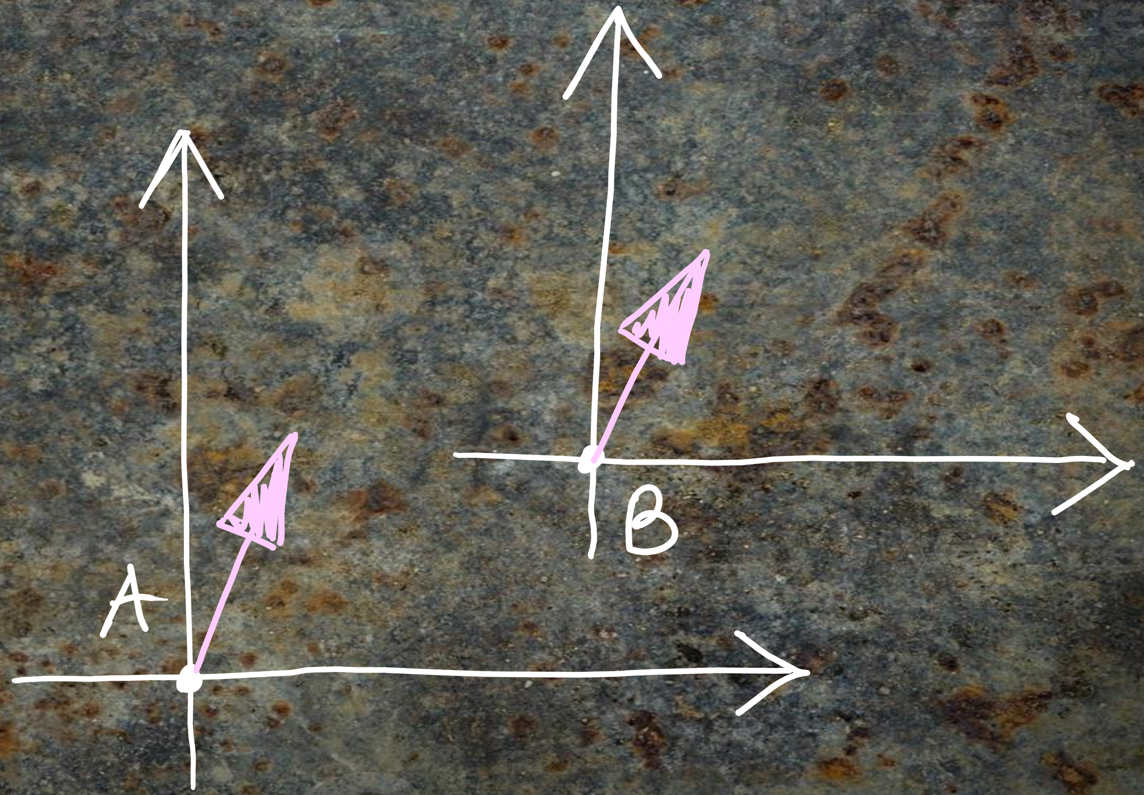

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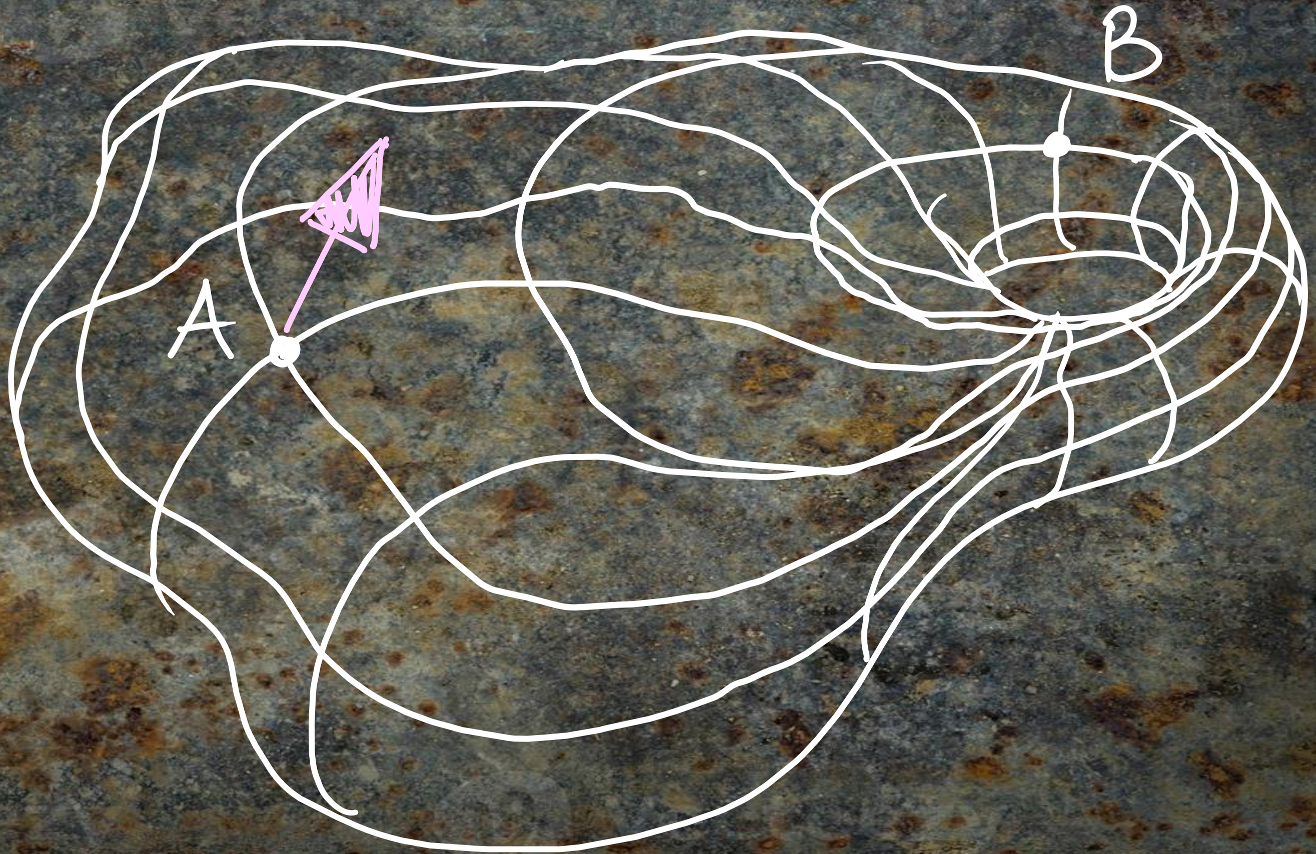

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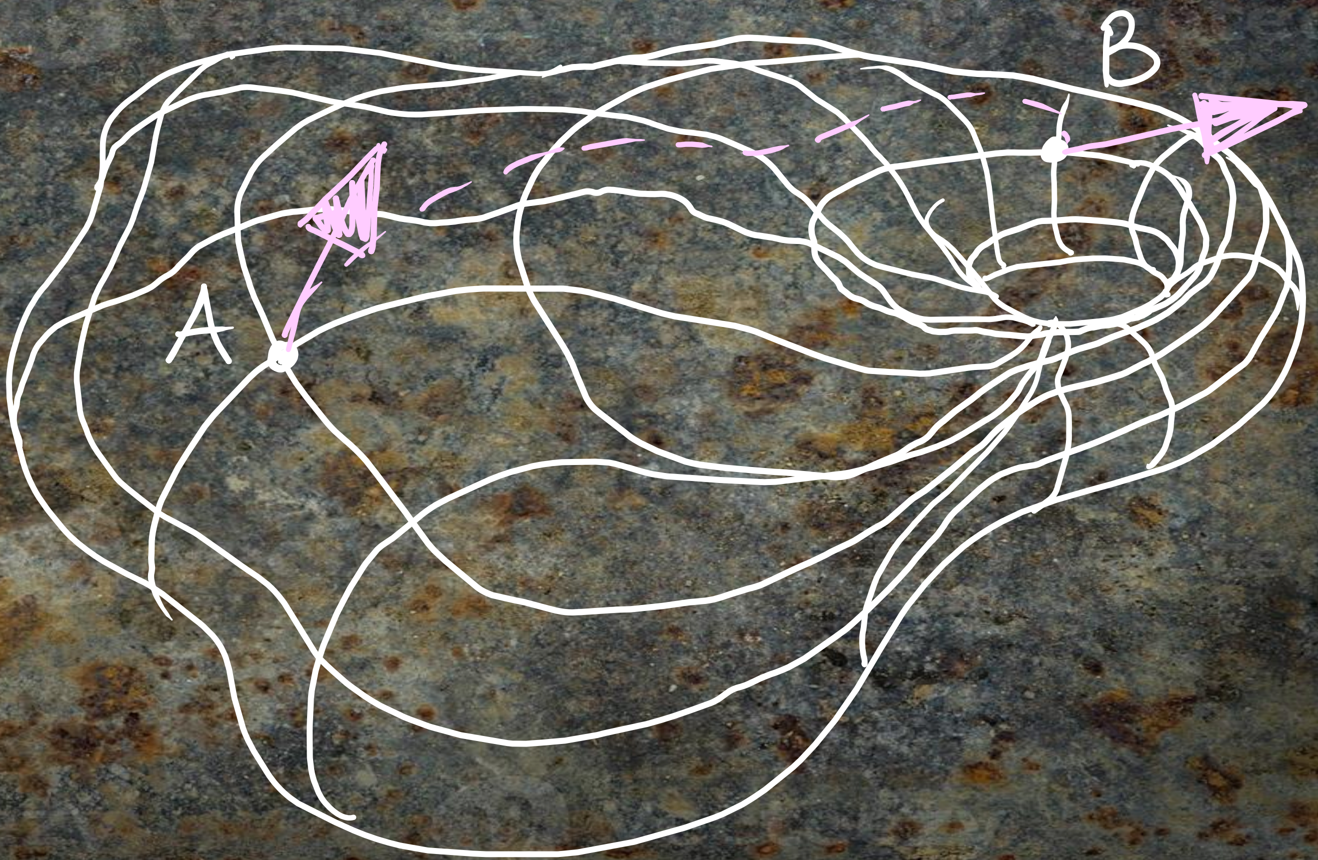

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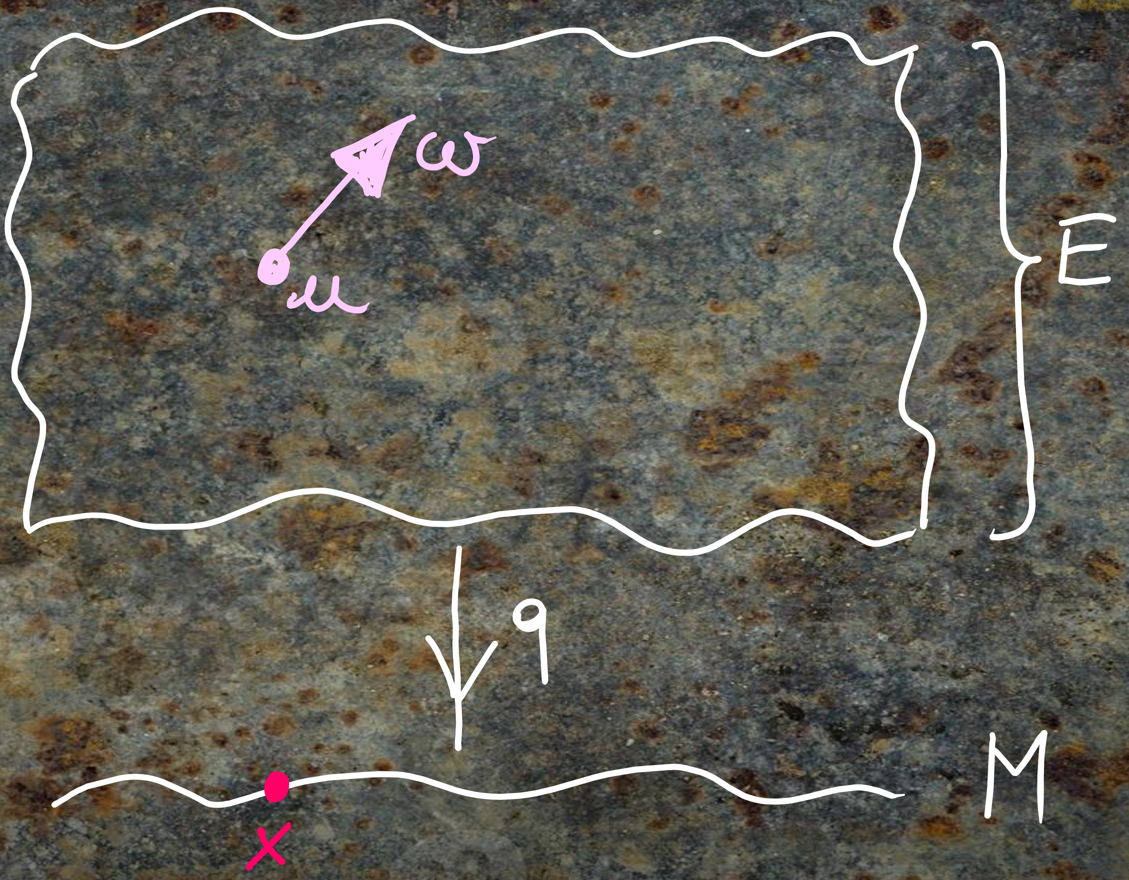
In a tangent category:

Take $q : E \to M$



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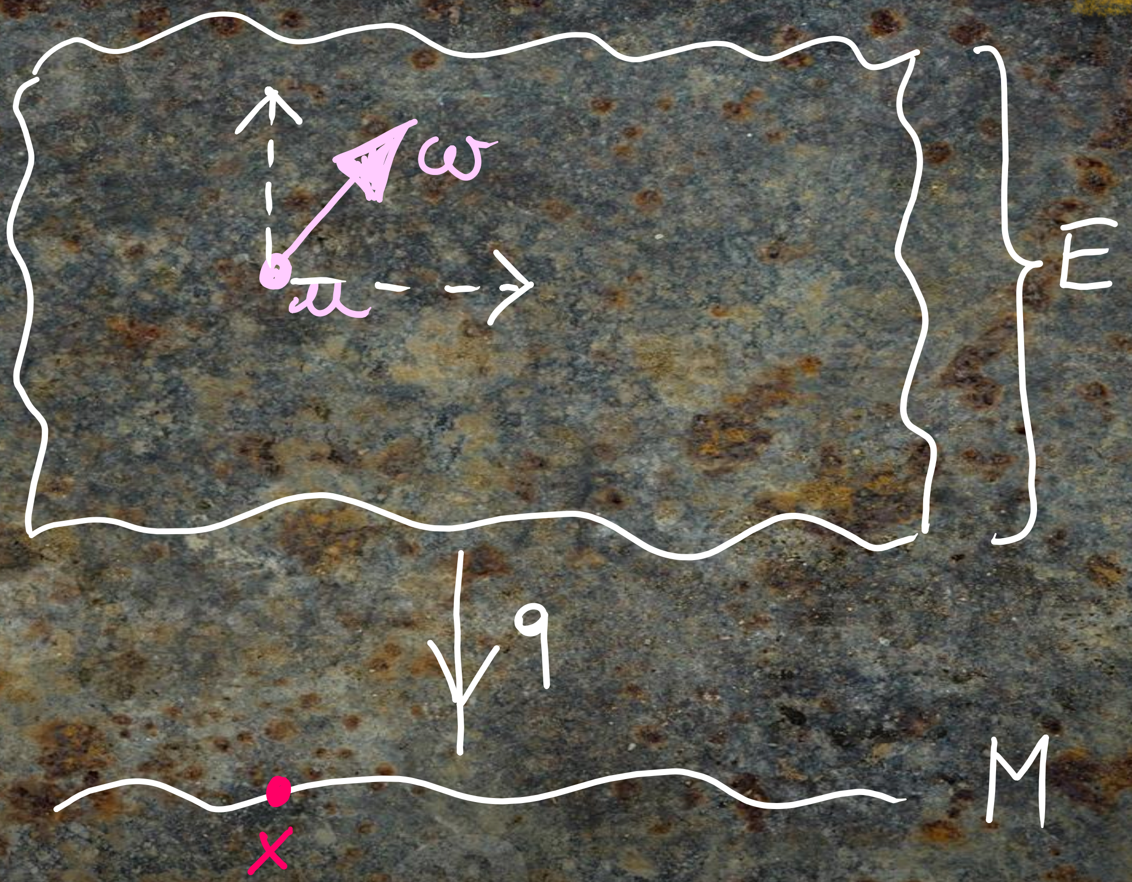
We want to introduce "coordinates" on E



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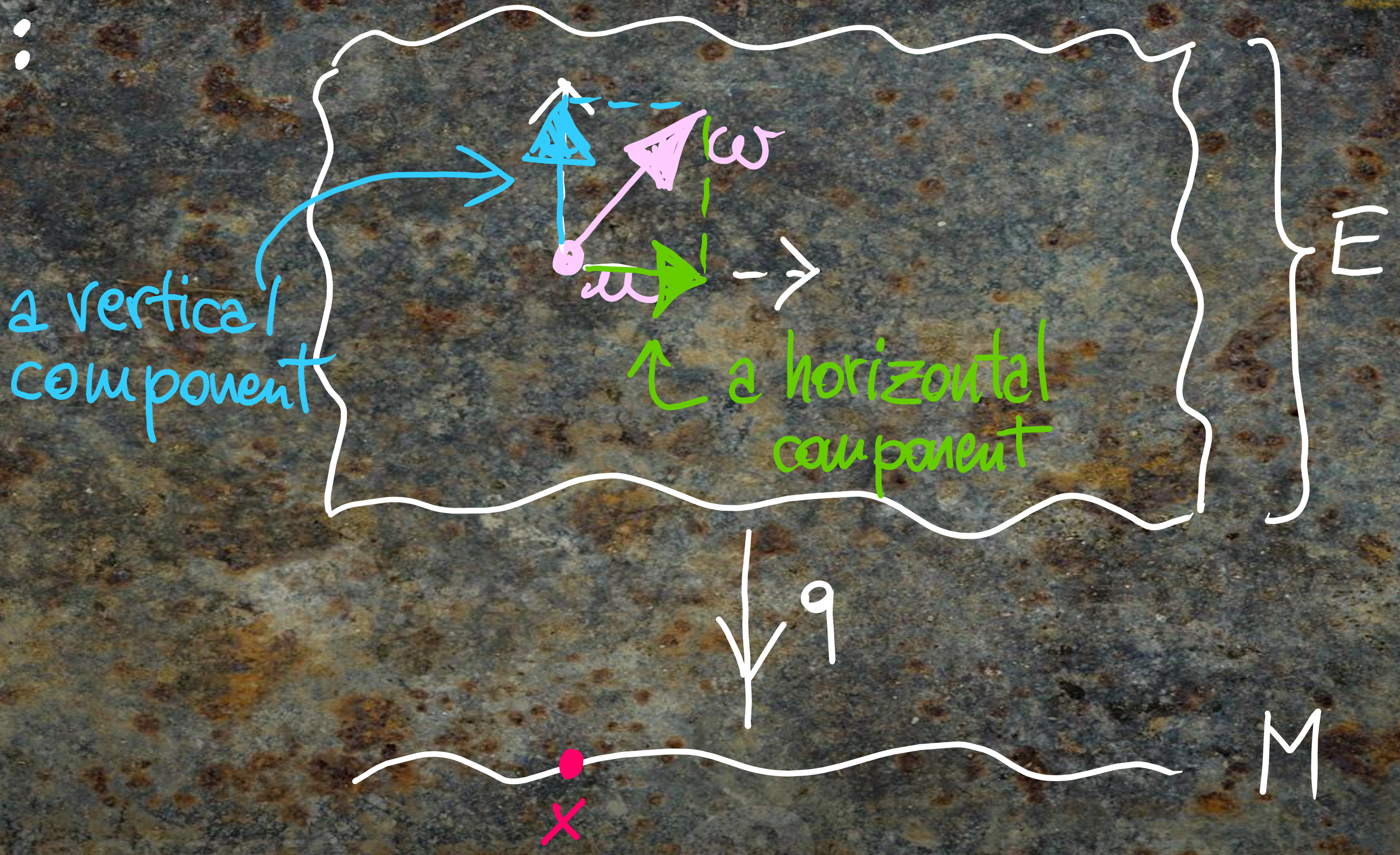

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Let's introduce the vertical & the horizontal bundles.

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$$\begin{array}{ccc} VE & \longrightarrow & TE \\ \downarrow & \lrcorner & \downarrow T_q \\ M & \xrightarrow[z]{} & TM \end{array} \qquad \begin{array}{ccc} HE & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow q \\ TM & \xrightarrow[p]{} & M \end{array}$$

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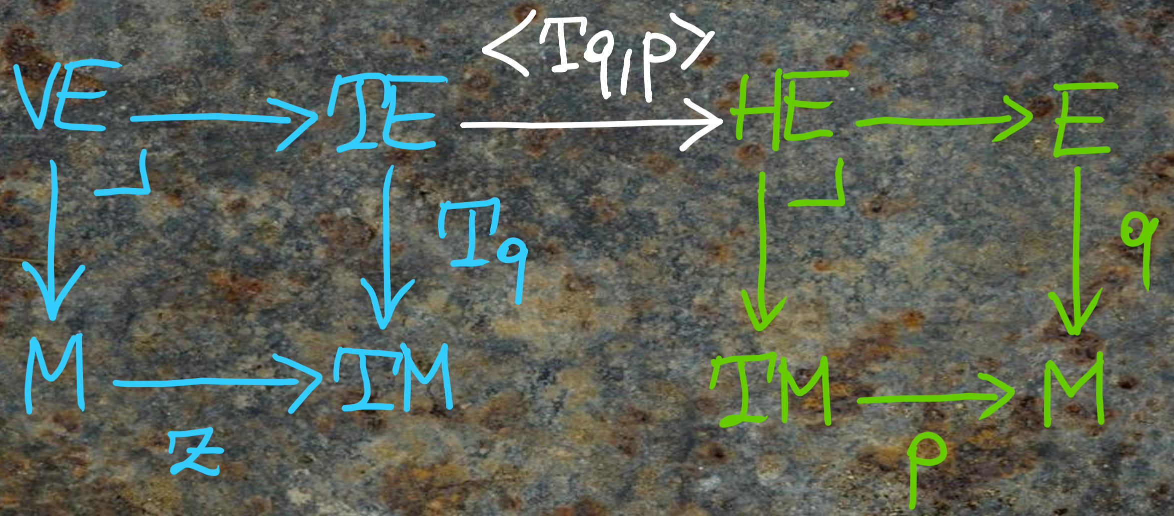

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We need that also $\mathbb{T}^{(M)} q$ has a vertical bundle!

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Cockett & Cruttwell's solution (2017)
(new version MacAdam (2022))

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Def'n.
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Example:

* submersions in differential geometry

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* It's extra structure
* Connections or the slice tang. cat. don't seem to depend on such a choice

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Let's start with display maps.

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A display map $q: E \to M$ in a category $\mathbb{X}$ is a morphism which admits pullbacks along any morphism.

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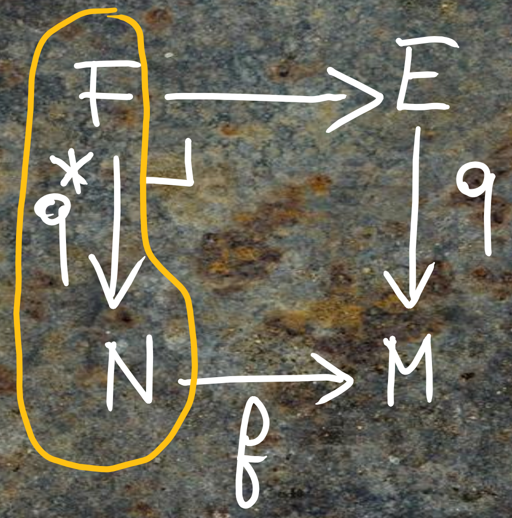
is q^* also a display map?



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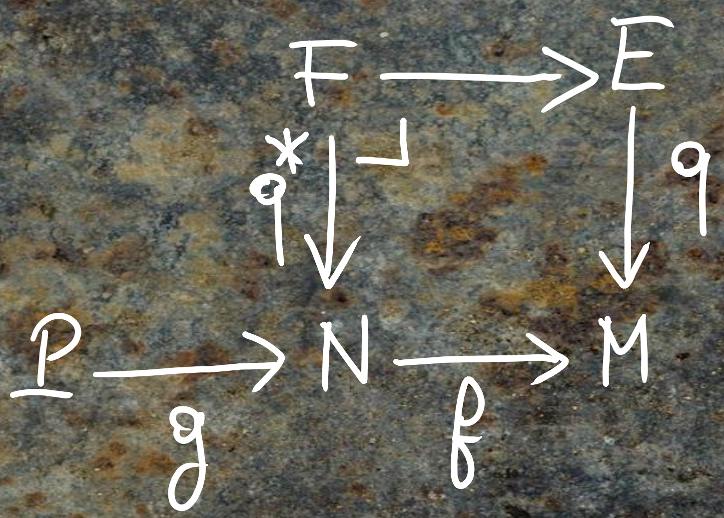

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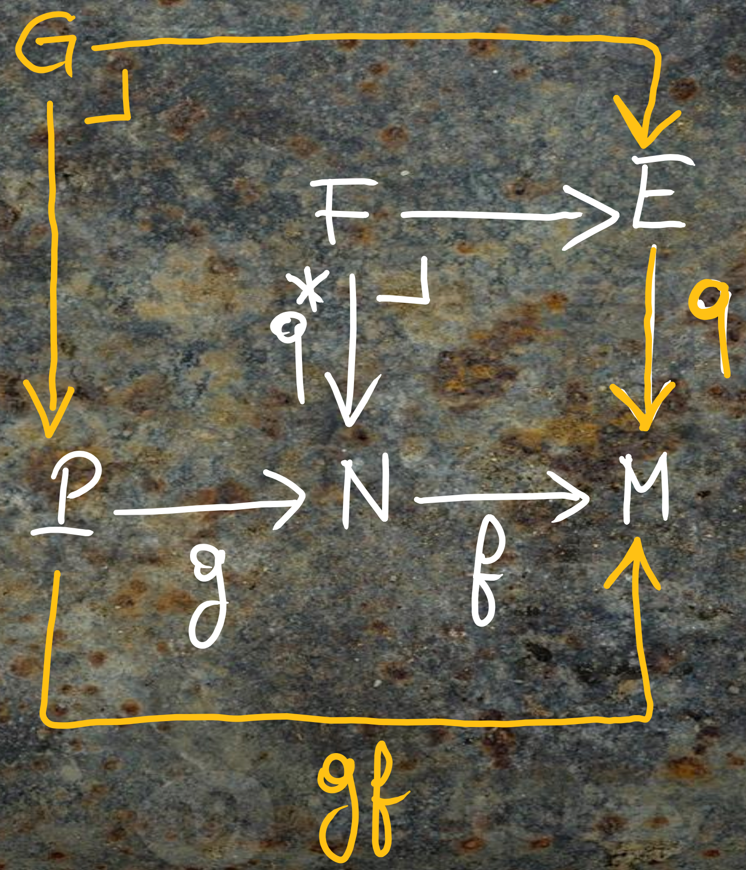

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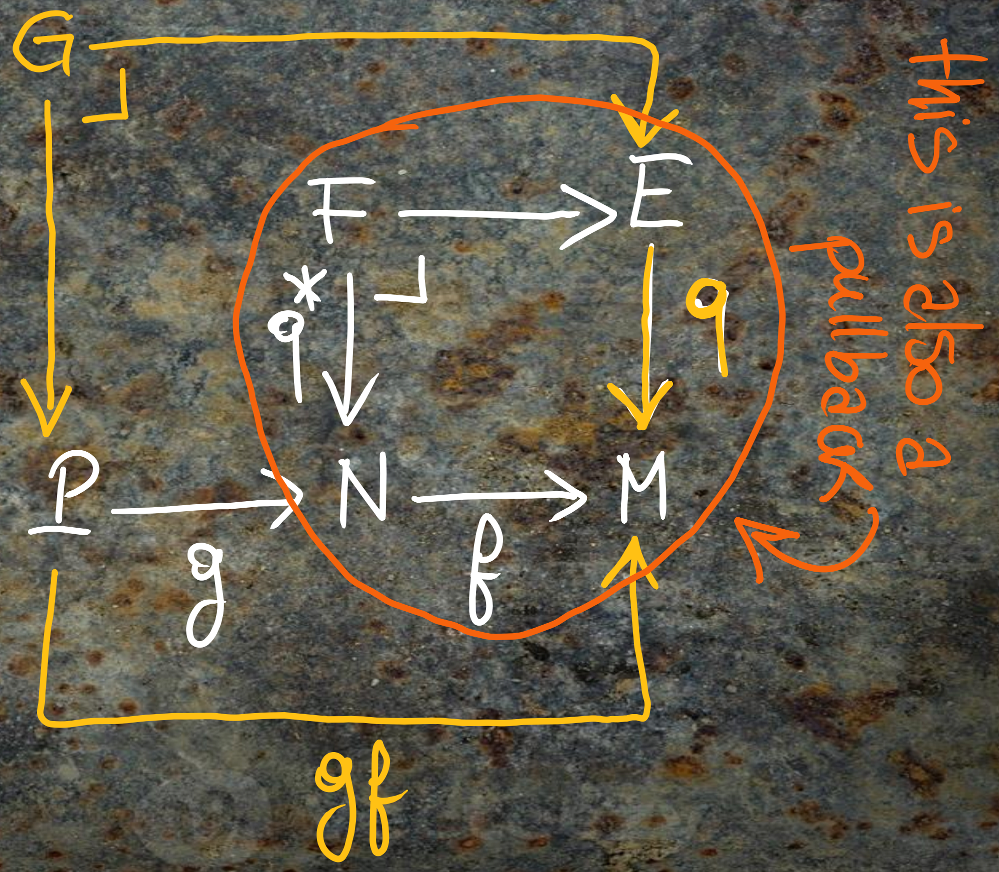

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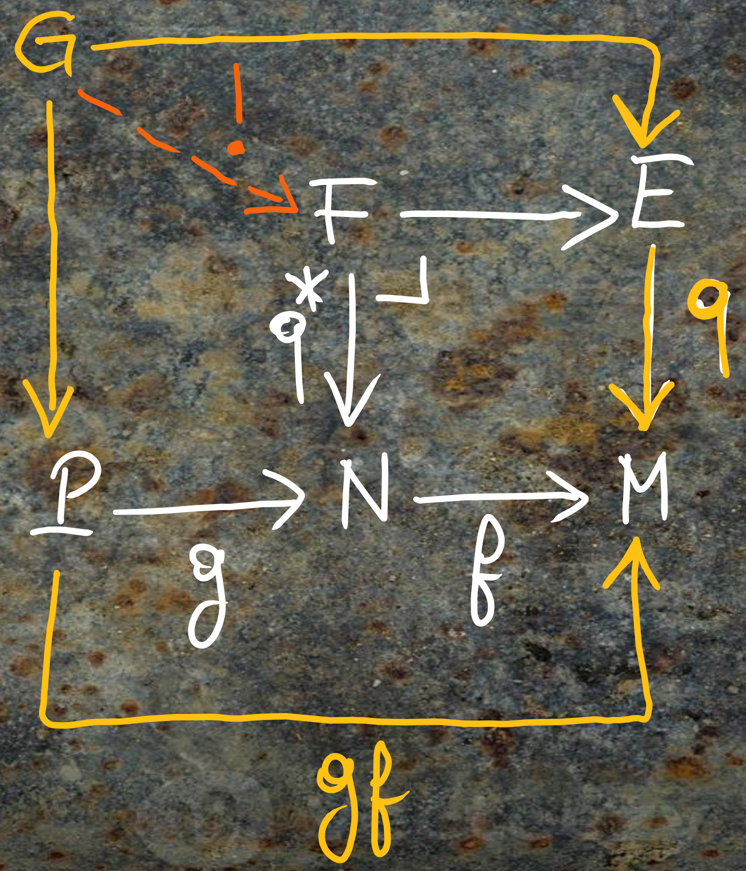

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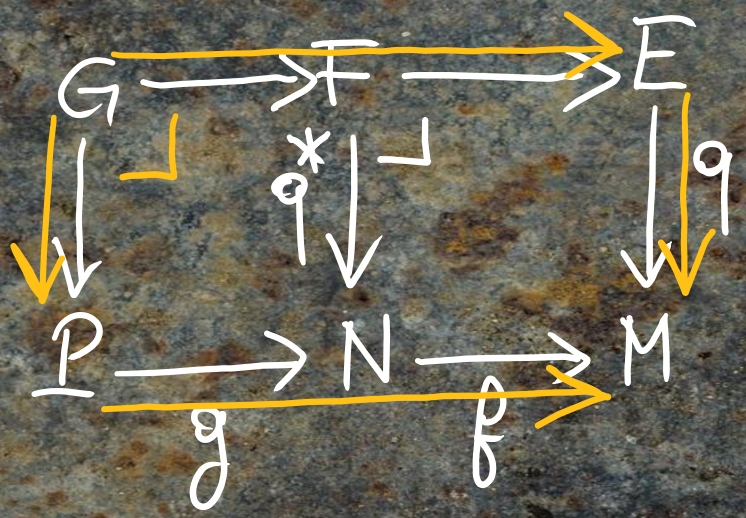

Tangent display maps

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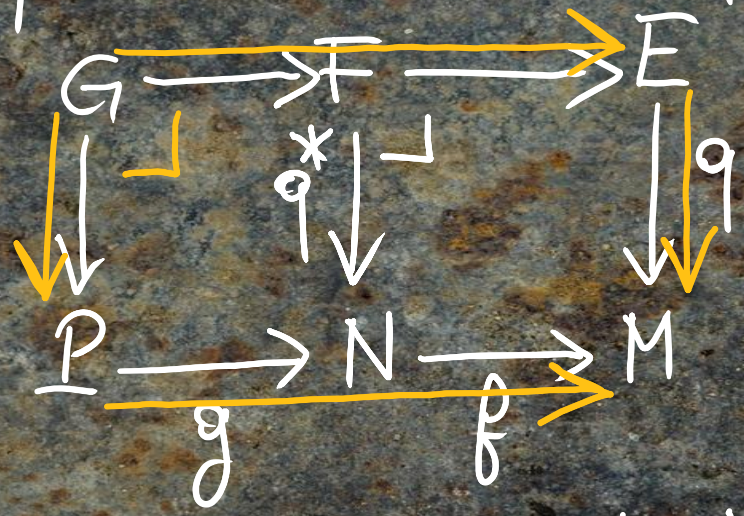
* If the right & outer squares are pullbacks so is the left square:



* Thus q^* is also a display map!

Tangent display maps

Lemma.
Let $\mathfrak{D}(\mathbb{X})$ denote the family of display maps of $\mathbb{X}$.
Then $\mathfrak{D}(\mathbb{X})$ is stable under pullbacks.

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Lemma.
Let $\mathcal{D}(\mathbb{X})$ denote the family of display maps of $\mathbb{X}$.
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In a category $\mathbb{X}$ equipped with an endofunctor $T : \mathbb{X} \to \mathbb{X}$ a T-display map $q : E \to M$ is a morphism for which $\forall m \in \mathbb{N}$ $T^m q : T^m E \to T^m M$ is display & $\forall m$ T^m preserves the pullback.

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* In a tangent category $(\mathbb{X}, \mathbb{T})$ T-display maps are called tangent display maps.

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$$\mathbb{T}^m\left(\begin{array}{ccc} P & \longrightarrow & \mathbb{T}^n E \\ \downarrow & \lrcorner & \downarrow \mathbb{T}^n_q \\ N & \xrightarrow[\forall f]{} & \mathbb{T}^n M \end{array}\right)$$

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Thm. (Cruttwell & Lanfranchi) Let $\mathcal{D}(\mathbb{X},\mathbb{T})$ denote the family of tangent display maps of a tangent category $(\mathbb{X},\mathbb{T})$.

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* $\mathcal{D}(\mathbb{X},\mathbb{T})$ is closed under composition.

What about differential bundles?

(Cockett & Cruttwell 2017)

Differential bundles are additive bundles whose vertical bundle is trivial.

$$E_2 \xrightarrow{s_q} E \underset{z_q}{\overset{q}{\rightleftarrows}} M$$

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A display tangent category is a tangent category whose tangent bundles are display diff. bundles.

A fully display tangent category is a tangent category whose diff. bundles are all display.

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\begin{array}{ccccc}
F & \xrightarrow{r} & E & \xrightarrow{s} & F \\
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q is a retraction of u

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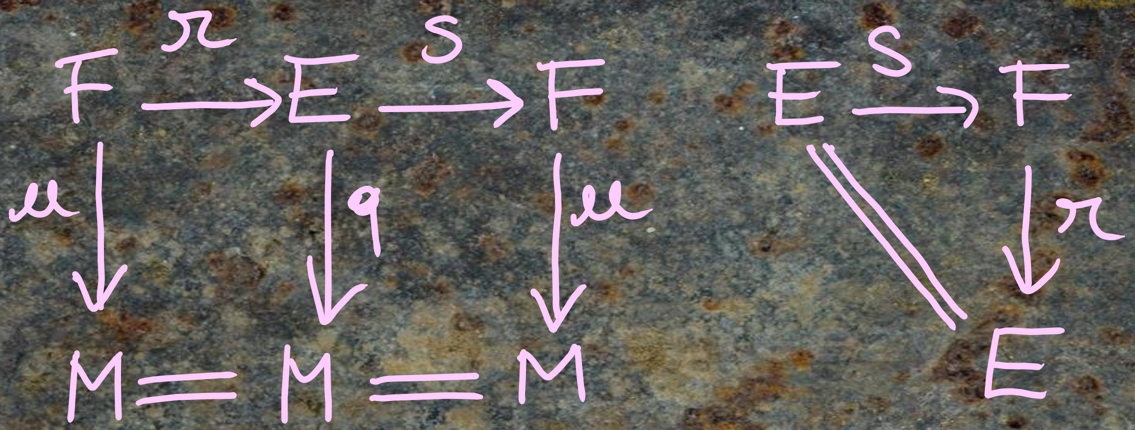

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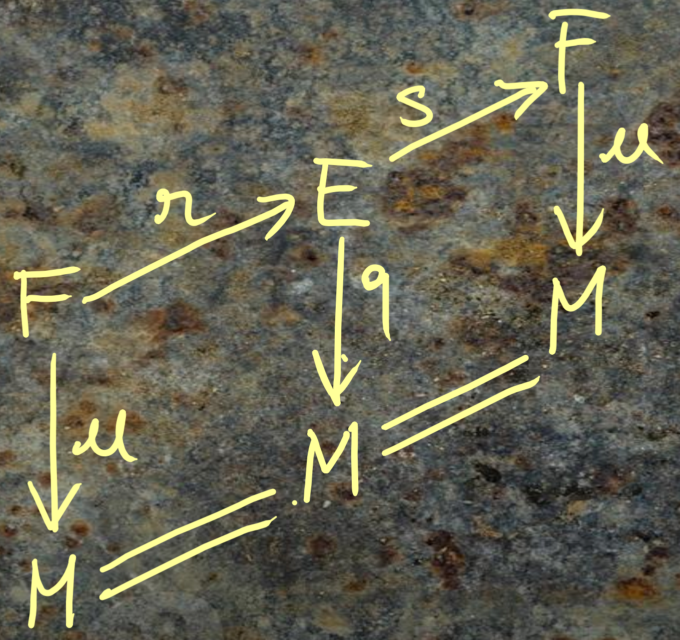

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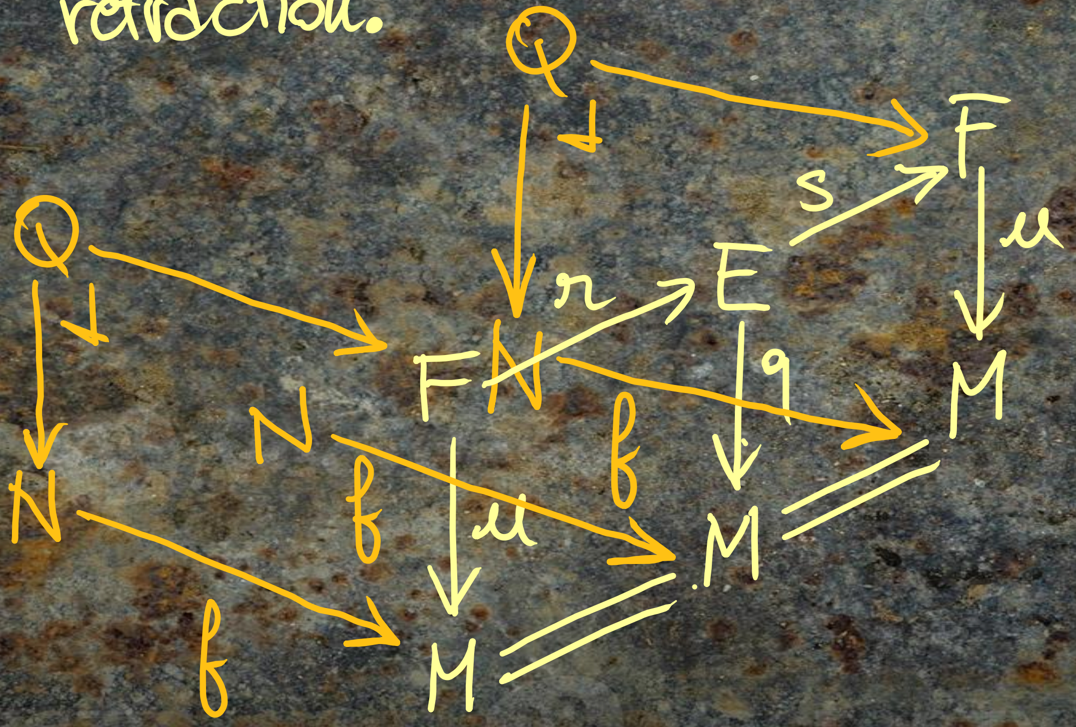

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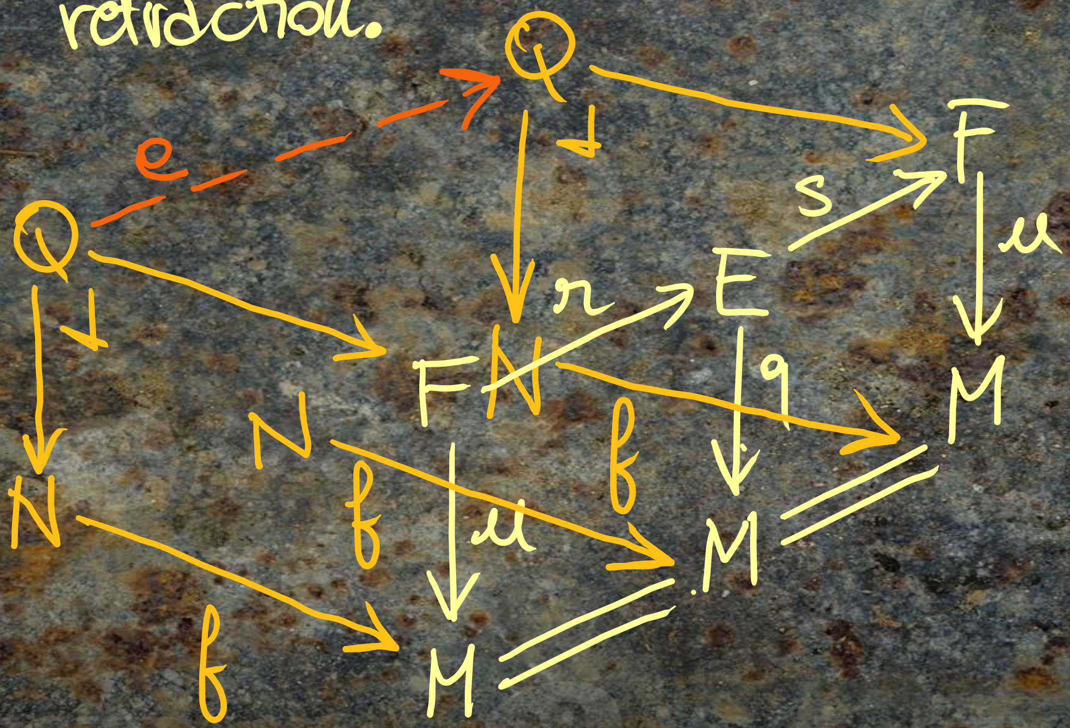

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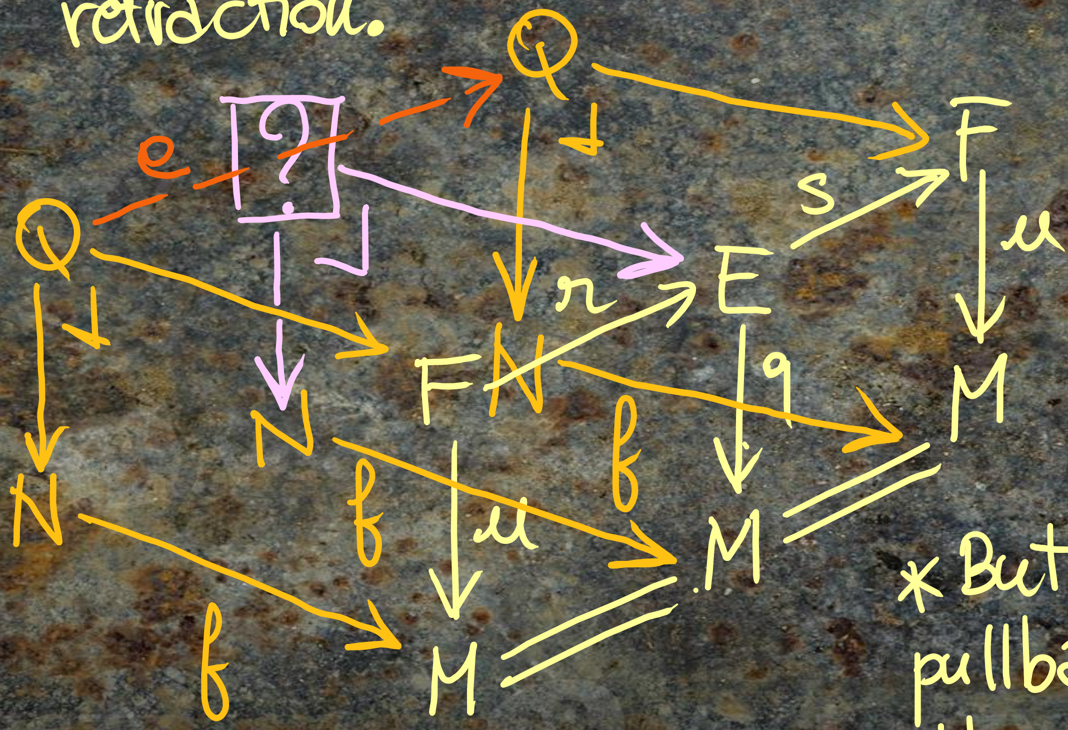

* But e is the pullback of an idempotent!..

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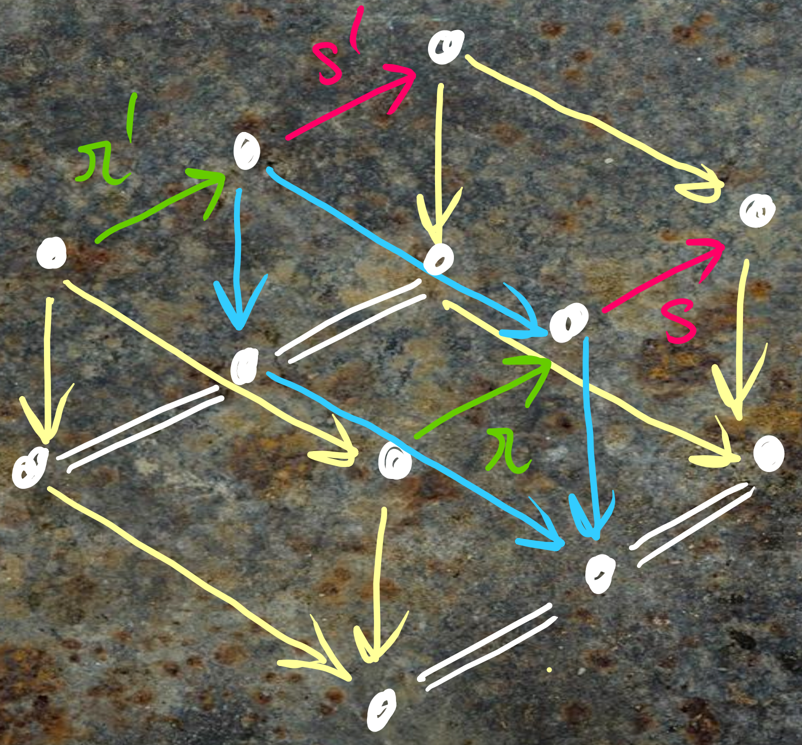
Suppose (s', π') & (s, π) are section-retraction pairs.



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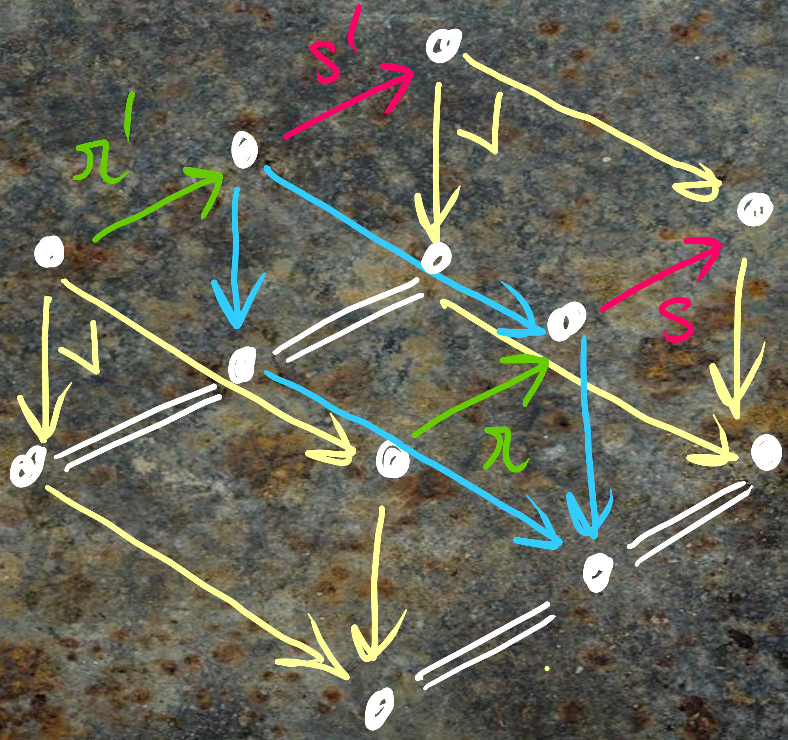

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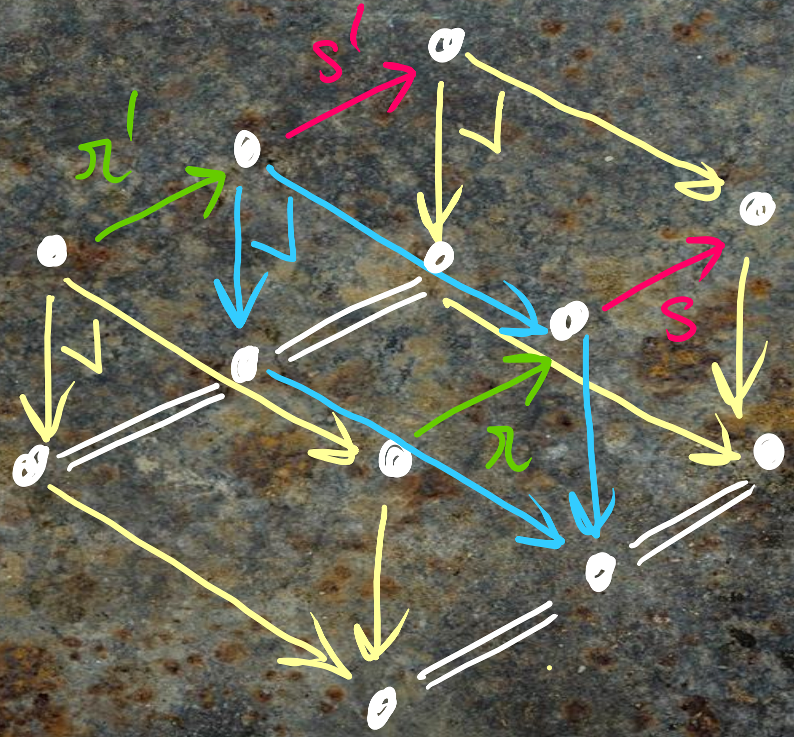

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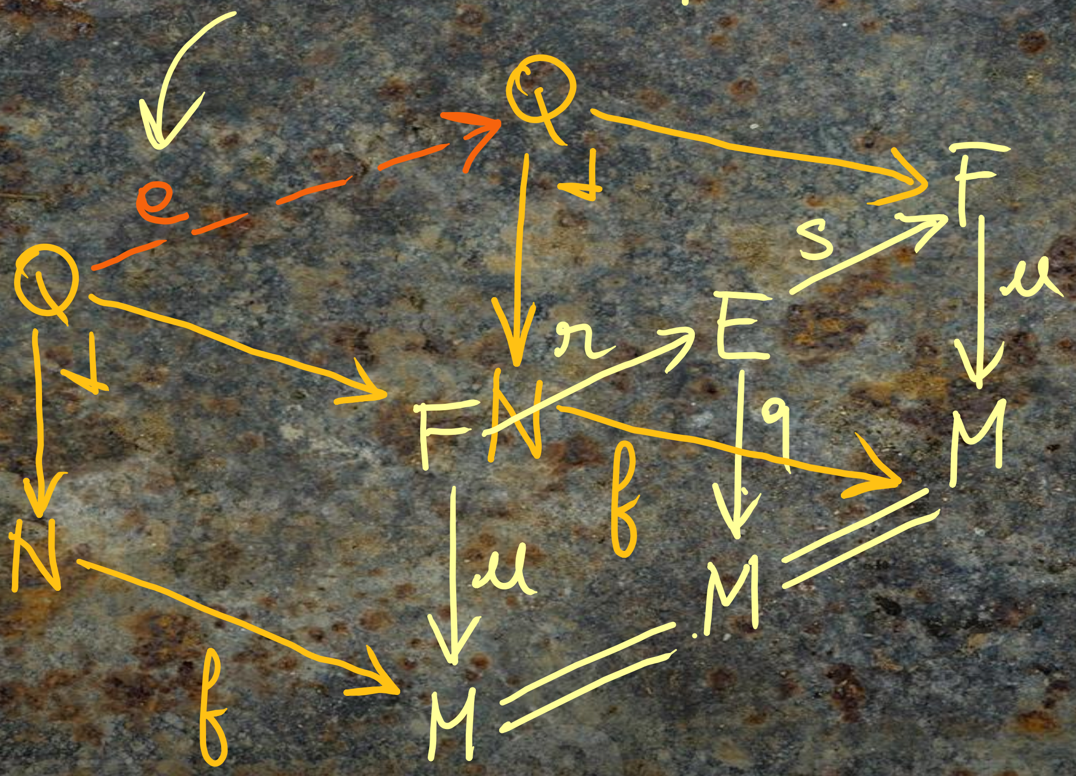

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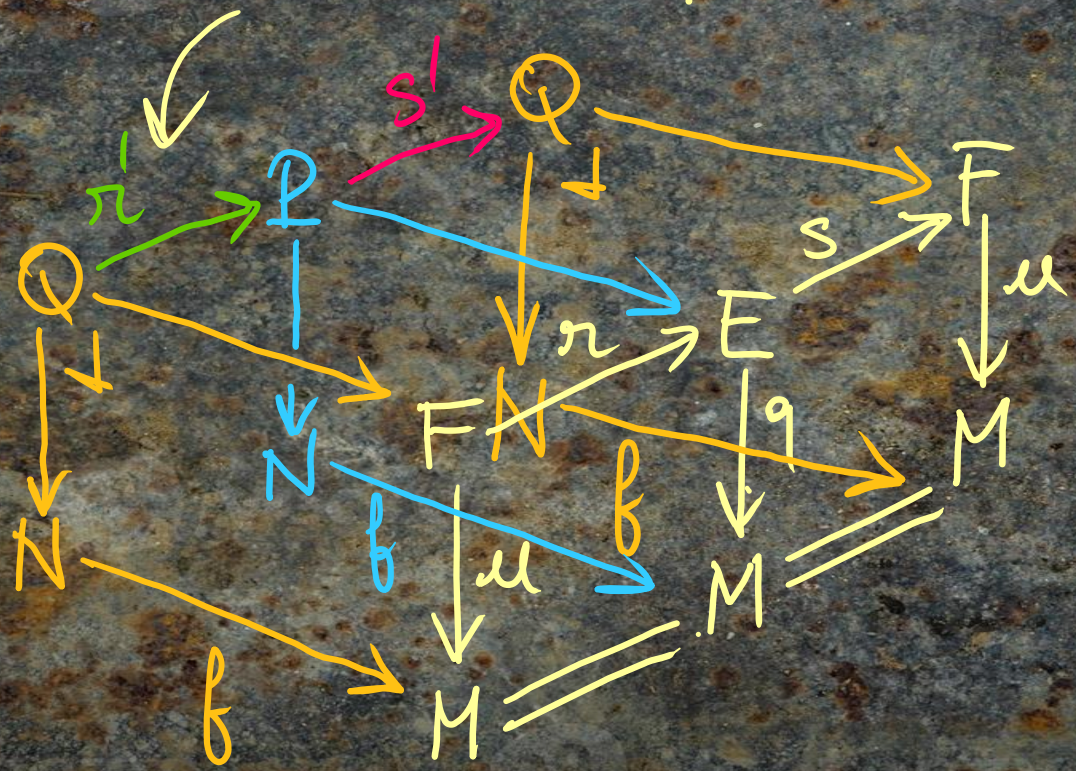

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Thm. (Cruttwell & Lanfranchi).
The Karoubi envelope of a tangent category (with negatives) is still a tangent category (with negatives).

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thanks.
* Cockett & Cruttwell (2017)
Differential bundles & fibrations for tangent categories
* MacAdam (2020)
Vector bundles & Differential bundles in the tangent category of smooth manifolds
* Cruttwell & Lemay (2023)
Differential bundles in commutative algebra & algebraic geometry
* Lanfranchi (2023)
The differential bundles in the geometric tang. cat. of an operad.